Cosmology 2 (Prof. Rennan Barkana): Solution to Homework 1

1. In the case without Λ, we can do this analytically (numerically is also OK). Letting Ω_m and Ω_r be the present Ω in matter and radiation, the Friedmann equation is:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = H^2 = H_0^2\left(\frac{\Omega_m}{a^3}+\frac{\Omega_r}{a^4}\right) ,$$

where $\Omega_r = 1-\Omega_m$. Solving this for dt, we get

$$t(a) = \int dt = H_0^{-1}\int_0^a \frac{da}{\sqrt{\Omega_m/a+\Omega_r/a^2}} = \frac{2}{3\Omega_m^2 H_0}\left[2\Omega_r^{3/2}+\sqrt{\Omega_m a+\Omega_r}\,(\Omega_m a-2\Omega_r)\right] .$$

Similarly,

$$\tau(a) = \int\frac{dt}{a} = \frac{2}{\Omega_m H_0}\left(\sqrt{\Omega_m a+\Omega_r}-\sqrt{\Omega_r}\right) .$$

With a cosmological constant, we have:

$$t(a) = \int dt = H_0^{-1}\int_0^a \frac{da}{\sqrt{\Omega_m/a+\Omega_r/a^2+\Omega_\Lambda a^2}} ,$$

and

$$\tau(a) = \int\frac{dt}{a} = H_0^{-1}\int_0^a \frac{da}{\sqrt{\Omega_m a+\Omega_r+\Omega_\Lambda a^4}} .$$

Also note that $H_0^{-1} = 9.78$ Gyr$/h = 14.4$ Gyr. I will use index 1 for the model without Λ, and 2 with Λ. Today $a=1$. In the first model, $\Omega_m = 1-9.07\times10^{-5}$, and in the second model, $\Omega_m = 1-0.689-9.07\times10^{-5}$. So matter-radiation equality, $a=\Omega_r/\Omega_m$ is different in the two models: it is $a=9.07\times10^{-5}$ in the first model and $a=2.92\times10^{-4}$ in the second. Other example redshifts: $z=8$ (roughly cosmic reionization) and $z=1100$ (cosmic recombination). The numerical values (in Gyr units) are $t_1(9.07\times10^{-5}) = 4.87\times10^{-6}$, $t_2(2.92\times10^{-4}) = 5.04\times10^{-5}$, $\tau_1(9.07\times10^{-5}) = 0.114$, $\tau_2(2.92\times10^{-4}) = 0.367$, $t_1(9.08\times10^{-4}) = 2.38\times10^{-4}$, $t_2(9.08\times10^{-4}) = 3.67\times10^{-4}$, $\tau_1(9.08\times10^{-4}) = 0.638$, $\tau_2(9.08\times10^{-4}) = 0.910$, $t_1(0.111) = 0.356$, $t_2(0.111) = 0.637$, $\tau_1(0.111) = 9.36$, $\tau_2(0.111) = 16.4$, $t_1(1) = 9.63$, $t_2(1) = 13.8$, $\tau_1(1) = 28.6$, $\tau_2(1) = 46.2$.

2. The smoothed density field is

$$\bar\delta(\vec{x}) = \int d^3x_1\, W(|\vec{x}_1-\vec{x}|)\delta(\vec{x}_1) .$$

Then

$$\sigma^2 = \left\langle \bar\delta(\vec{x})\,\bar\delta(\vec{x})\right\rangle = \int d^3x_1 \int d^3x_2\, W(|\vec{x}_1-\vec{x}|)W(|\vec{x}_2-\vec{x}|)\,\langle\delta(\vec{x}_1)\delta(\vec{x}_2)\rangle\ .$$

Using inverse Fourier transforms and the definition of the power spectrum, we showed in class that

$$\langle\delta(\vec{x}_1)\delta(\vec{x}_2)\rangle = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}P(k)\ .$$

We use this in the expression for σ^2, and also we write each term of the form $W(r)$ as the inverse Fourier transform of $\tilde{W}(k)$. We get an expression with five integrals:

$$\sigma^2 = \int d^3x_1 \int d^3x_2 \int \frac{d^3k}{(2\pi)^3} \int d^3k_1 \int d^3k_2\, e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}P(k)\,\frac{\tilde{W}(k_1)}{(2\pi)^3}e^{i\vec{k}_1\cdot(\vec{x}_1-\vec{x})}\,\frac{\tilde{W}(k_2)}{(2\pi)^3}e^{i\vec{k}_2\cdot(\vec{x}_2-\vec{x})}\ .$$

The integral over $\vec{x}_1$ gives $(2\pi)^3$ times a Dirac delta function of $\vec{k}+\vec{k}_1$, and then the $\vec{k}_1$ integral simply sets $\vec{k}_1$ equal to $-\vec{k}$. We similarly evaluate the $\vec{x}_2$ and $\vec{k}_2$ integrals (i.e., we set $\vec{k}_2$ equal to $\vec{k}$). Thus, we obtain (note that $\tilde{W}(k)$ only depends on the magnitude of $\vec{k}$):

$$\sigma^2 = \frac{1}{(2\pi)^3}\int d^3k\,\tilde{W}^2(k)P(k)\ .$$

Note that the result does not depend on the starting point $\vec{x}$ (since this field is statistically homogeneous).

3. The variance is:

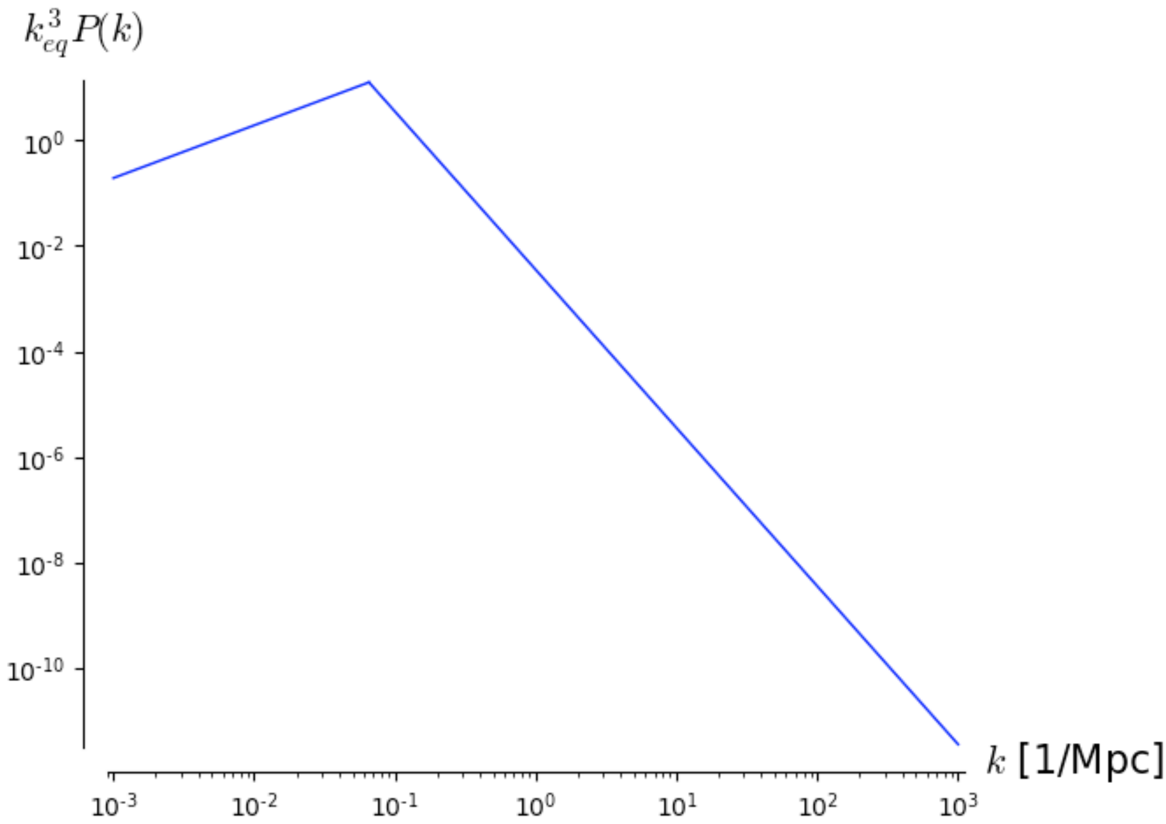
$$\sigma^2(R) = \int_0^\infty \frac{1}{2\pi^2}k^2dkP(k)\frac{9}{x^6}[\sin(x)-x\cos(x)]^2\ ,$$

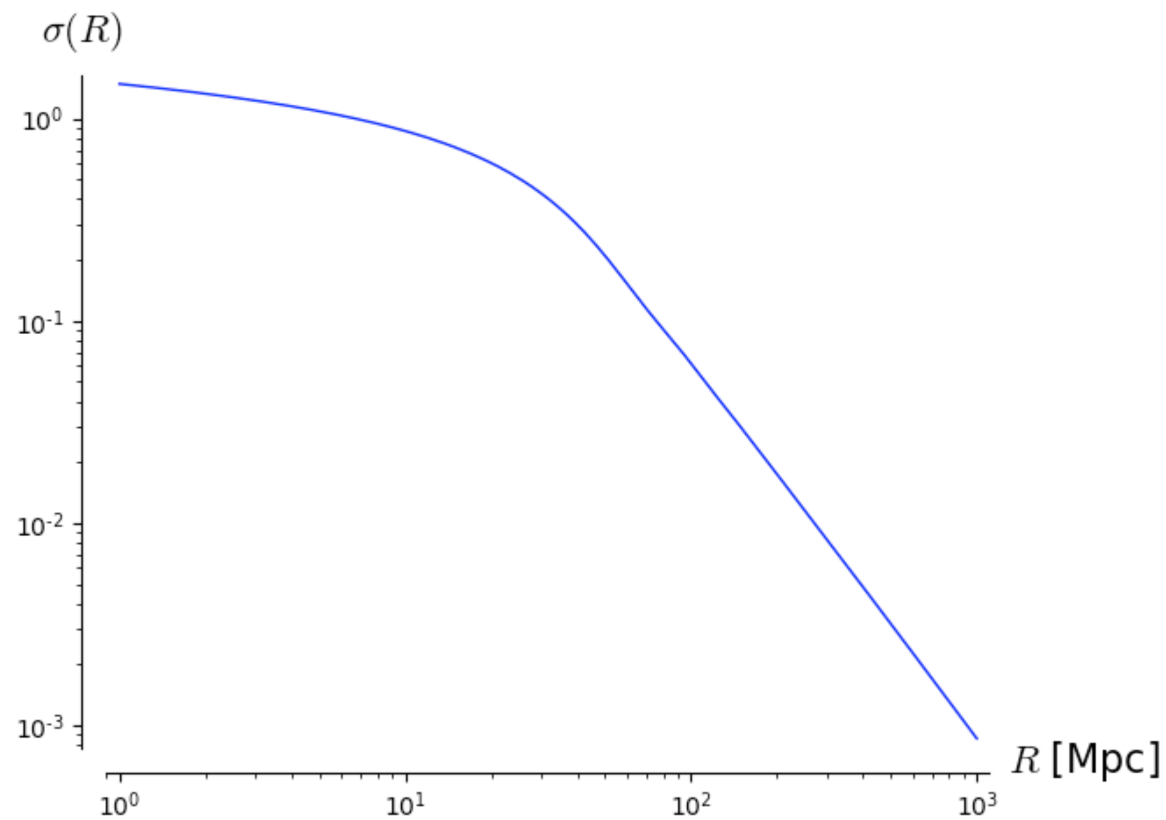
where $x=kR$ and $P(k)=Ak/k_{\rm eq}$ for $k<k_{\rm eq}$, $P(k)=A[k/k_{\rm eq}]^{-3}$ for $k>k_{\rm eq}$. Thus,

$$\sigma^2(R) = \frac{9}{2\pi^2}Ak_{\rm eq}^3\left\{\frac{1}{x_{\rm eq}^4}\int_0^{x_{\rm eq}}\frac{dx}{x^3}[\sin(x)-x\cos(x)]^2 + \int_{x_{\rm eq}}^\infty \frac{dx}{x^7}[\sin(x)-x\cos(x)]^2\right\}\ ,$$

where $x_{\rm eq}=k_{\rm eq}R$.

We calculate that $k_{\rm eq}8/h = 0.772$, and then the normalization to $\sigma(R_0)=0.81$ gives $Ak_{\rm eq}^3 = 12.6$. See the plots below, of the dimensionless quantities $P(k)k_{\rm eq}^3$ and $\sigma(R)$.


$k_{eq}^3 P(k)$
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
10^2
10^3
k [1/Mpc]


$\sigma(R)$
10^0
10^{-1}
10^{-2}
10^{-3}
10^0
10^1
10^2
10^3
R [Mpc]