Cosmology 2 (Prof. Rennan Barkana): Solution to Homework 1

1. a. Letting Ω_m and Ω_r be the present Ω in matter and radiation, the Friedmann equation is:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = H^2 = H_0^2\left(\frac{\Omega_m}{a^3}+\frac{\Omega_r}{a^4}\right) ,$$

where $\Omega_r = 1-\Omega_m$. Solving this for dt, we get

$$t(a) = \int dt = H_0^{-1}\int_0^a \frac{da}{\sqrt{\Omega_m/a+\Omega_r/a^2}} = \frac{2}{3\Omega_m^2 H_0}\left[2\Omega_r^{3/2}+\sqrt{\Omega_m a+\Omega_r}\,(\Omega_m a-2\Omega_r)\right] .$$

Similarly,

$$\tau(a) = \int\frac{dt}{a} = \frac{2}{\Omega_m H_0}\left(\sqrt{\Omega_m a+\Omega_r}-\sqrt{\Omega_r}\right) .$$

1. b. With a cosmological constant, we have:

$$t(a) = \int dt = H_0^{-1}\int_0^a \frac{da}{\sqrt{\Omega_m/a+\Omega_r/a^2+\Omega_\Lambda a^2}} ,$$

and

$$\tau(a) = \int\frac{dt}{a} = H_0^{-1}\int_0^a \frac{da}{\sqrt{\Omega_m a+\Omega_r+\Omega_\Lambda a^4}} .$$

Also note that $H_0^{-1} = 9.78$ Gyr/h = 14.4 Gyr. I will use index 1 for the model with Λ, and 2 without Λ. The numerical values (in Gyr units) are $t_1(10^{-4}) = 6.86\times10^{-6}$, $t_2(10^{-4}) = 5.81\times10^{-6}$, $\tau_1(10^{-4}) = 0.141$, $\tau_2(10^{-4}) = 0.124$, $t_1(0.5) = 5.84$, $t_2(0.5) = 3.39$, $\tau_1(0.5) = 35.1$, $\tau_2(0.5) = 20.1$, $t_1(1) = 13.8$, $t_2(1) = 9.60$, $\tau_1(1) = 46.1$, $\tau_2(1) = 28.5$. See these quantities plotted in the figure.

2. a. We take the inverse transform of $\hat{\delta}(\vec{k})$, and check that we indeed get $\delta(\vec{x})$ back:

$$\int d^3k\, e^{i\vec{k}\cdot\vec{x}}\int\frac{d^3x'}{(2\pi)^3}\,e^{-i\vec{k}\cdot\vec{x}'}\,\delta(\vec{x}') = \int\frac{d^3x'}{(2\pi)^3}\,\delta(\vec{x}')\,(2\pi)^3\delta_D\left(\vec{x}-\vec{x}'\right) = \delta(\vec{x}) ,$$

where the integration over $\vec{k}$ yielded a Dirac Delta function.

2. b. The smoothed density field is

$$\bar{\delta}(\vec{x}) = \int d^3x_1\, W(|\vec{x}_1-\vec{x}|)\delta(\vec{x}_1) .$$

Then

$$\sigma^2 = \left\langle \bar{\delta}(\vec{x})\, \bar{\delta}(\vec{x}) \right\rangle = \int d^3x_1 \int d^3x_2\, W(|\vec{x}_1 - \vec{x}|) W(|\vec{x}_2 - \vec{x}|)\, \langle \delta(\vec{x}_1)\delta(\vec{x}_2) \rangle \ .$$

Using inverse Fourier transforms and the definition of the power spectrum, we showed in class that

$$\langle \delta(\vec{x}_1)\delta(\vec{x}_2) \rangle = \int d^3k\, e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)} P(k) \ .$$

We use this in the expression for σ^2, and also we write each term of the form $W(r)$ as the Fourier transform of $\tilde{W}(k)$. We get an expression with five integrals:

$$\sigma^2 = \int d^3x_1 \int d^3x_2 \int d^3k \int d^3k_1 \int d^3k_2\, e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)} P(k)\, \frac{\tilde{W}(k_1)}{(2\pi)^3} e^{-i\vec{k}_1\cdot(\vec{x}_1 - \vec{x})}\, \frac{\tilde{W}(k_2)}{(2\pi)^3} e^{-i\vec{k}_2\cdot(\vec{x}_2 - \vec{x})} \ .$$

The integral over $\vec{x}_1$ gives $(2\pi)^3$ times a Dirac delta function of $\vec{k} - \vec{k}_1$, and then the $\vec{k}_1$ integral simply sets $\vec{k}_1$ equal to $\vec{k}$. We similarly evaluate the $\vec{x}_2$ and $\vec{k}_2$ integrals (i.e., we set $\vec{k}_2$ equal to $-\vec{k}$). Thus, we obtain (note that $\tilde{W}$ is real)

$$\sigma^2 = \int d^3k\, \tilde{W}^2(k) P(k) \ .$$

Note that the result does not depend on the starting point $\vec{x}$ (since this field is statistically homogeneous).

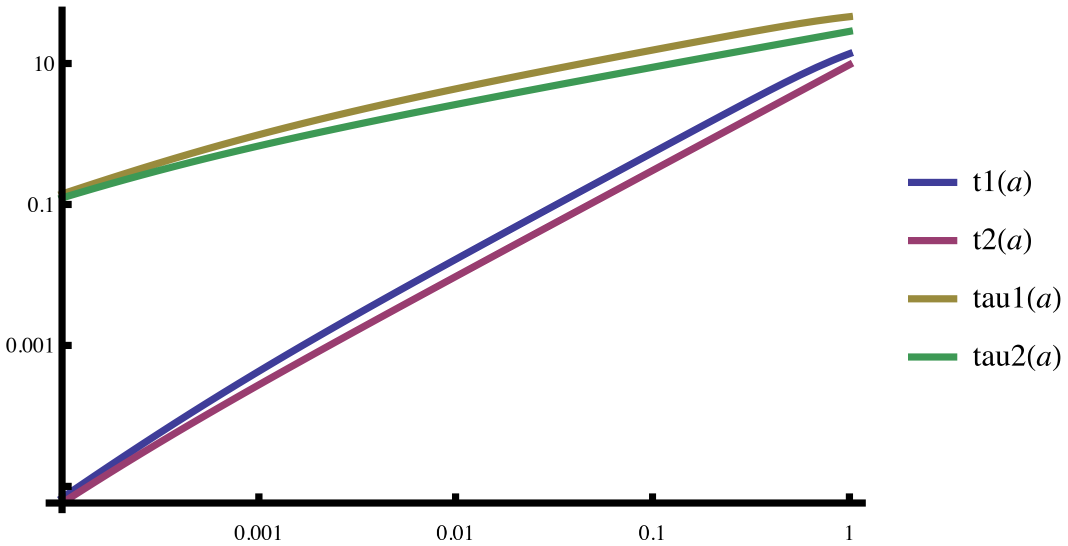

Fig. 1.— Various cosmic times versus the scale factor a. Shown are the cosmic age t_1 (with Λ) or t_2 (without Λ), and the comoving time τ_1 (with Λ) or τ_2 (without Λ).