TEL AVIV UNIVERSITY אוניברסיטת תל-אביב
RAYMOND AND BEVERLY SACKLER הפקולטה למדעים מדויקים
FACULTY OF EXACT SCIENCES ע"ש ריימונד ובברלי סאקלר
SCHOOL OF PHYSICS & ASTRONOMY בית הספר לפיסיקה ואסטרונומיה

The exam is three hours long.
No support materials are allowed at the exam.

Instructions: Answer the first question (30 points) and choose 2 of the next 3 (35 points each). Explain each step briefly (e.g., avoid writing a series of mathematical formulas with no explanation).

Formulas that you may use without explanation if you find them helpful:

$$k(R,t) = \frac{\partial f(R,t)}{\partial R} \qquad (m\Omega - \omega)^2 = \kappa^2 - 2\pi G\Sigma|k| + k^2 c_s^2$$

$$V_c^2 = \frac{GM(R)}{R} \qquad \beta = \theta - \alpha \qquad \gamma = \frac{4GM(\xi)}{\xi c^2} \qquad \mu = \left|\frac{\theta d\theta}{\beta d\beta}\right|$$

$$\Sigma_1(R,\phi,t) = H(R,t)e^{i[m\phi + f(R,t)]} \qquad \frac{\omega}{m} = \Omega \pm \frac{\kappa}{m}$$

$$\hat{\delta}(\vec{k}) \equiv \int \frac{d^3x}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \delta(\vec{x}) \qquad \sigma^2 = \int d^3k\, \tilde{W}(\vec{k})\tilde{W}(-\vec{k})P(\vec{k})$$

$$\delta(\vec{x}) = \int d^3k\, e^{i\vec{k}\cdot\vec{x}}\, \hat{\delta}(\vec{k}) \qquad \kappa^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2$$

$$\tilde{W}_{TH}(\vec{k}) = \frac{3}{(kR)^3}\left[\sin(kR) - kR\cos(kR)\right]$$

$$\delta^{\mathrm{D}}(\vec{k}) = \int \frac{d^3x}{(2\pi)^3} e^{\pm i\vec{k}\cdot\vec{x}} \qquad \xi(r) = 4\pi \int_0^\infty k^2 dk\, P(k)\, \frac{\sin(kr)}{kr}$$

<u>Question 1: Required</u>

In this question, write simple equations or approximate expressions (Include clear explanations, but no need for full mathematical derivations).

a. In spiral structure, what is the winding problem? What is the tight-winding approximation?
[10 points]

b. What is a singular isothermal sphere? Why is it important in astrophysics, and what objects does it describe well?
[10 points]

c. What is the Press-Schechter model? Describe its assumptions and results.
[10 points]

Question 2: Spiral arms

Consider spiral structure within a thin gas disk with a surface density: $\Sigma \propto R^{-n}$, where R is the two-dimensional radial distance in the disk. Assume that $0<n<2$. You may assume the tight-winding approximation.

a. Find $\Omega(R)$ and $\kappa(R)$.
[10 points]

b. Find the radii of the Lindblad resonances and of the corotation resonance, for a given wave solution with temporal frequency ω and m spiral arms.
[10 points]

c. Find the dispersion relation for this disk, at $R>0$, assuming $\omega \ll \Omega(R)$ and that the sound speed $c_S \ll R^*\Omega(R)$. What values of m can be described with your solution? Find the equation (in polar coordinates) describing the shape of a spiral arm for the case m=1.
[10 points]

d. Analyze the stability of this disk to axisymmetric perturbations. (Here, do **not** assume $\omega \ll \Omega$ or $c_S \ll R^*\Omega(R)$.) What is the condition for having more stability at large radii than at small radii?
[5 points]

Question 3: The power spectrum and correlation function

a. Write down a broken power-law approximation (i.e., including two different power-law segments) for the shape of the density power spectrum in the universe at present. Explain (roughly, not in full quantitative detail) what (in terms of cosmic history) sets the peak position (i.e., wavenumber) k_{peak} .
[10 points]

b. Write down a full expression for the correlation function corresponding to the power spectrum from part a. Also, find the same for the specific distance: $r=1/k_{peak}$. (No need to evaluate any integrals)
[10 points]

c. Evaluate the following expectation value:
$\left\langle \hat{\delta}(\vec{k}_1)\hat{\delta}(\vec{k}_2)\right\rangle$, where $\hat{\delta}$ is the Fourier transform of the density perturbation. Show that the result can be written in the form:
$P(k_1)\delta^{\mathrm{D}}(\vec{k}_1 + \vec{k}_2)$ (in terms of a Dirac Delta function), and find $P(k_1)$ in terms of the correlation function.
Hint: Once you obtain an expression that includes spatial position variables $\vec{x}_1, \vec{x}_2$, change variables to their mean and their difference.
[15 points]

Question 4: Gravitational lensing

Assume a lensing mass distribution with a surface density: $\Sigma \propto \xi^{-n}$, where ξ is the projected distance in the lens plane. Assume that $0<n<2$.

a. Find the Einstein angle in this case, and then write the lens equation where the source and lens positions are expressed in units of the Einstein angle.
[15 points]

b. Choose $n=1/2$. Make a plot of the lens equation (Your plot does not need to be exactly accurate, you only need to get the right general shape and the main features). Use the plot to find the **number** of images as a function of the source position.
[10 points]

c. Find the expression for the magnification of images, for the lens from part b. Find the caustics (i.e., source positions for which the magnification is infinite). What is the shape (projected on the sky) of the caustics?
[10 points]