In the notes, we skipped from the bottom of pg. 25 (“Special case”) to the middle of pg. 27 (“Moments of CBE”). This material is not needed.

The part from pg. 29 middle (“Comoving coordinates”) to pg. 31 middle (“Comments”) is repeated in the “Comoving Addition”, with the addition of other homogeneous components (in addition to matter). The added section also includes the full derivation of the Euler equation, which is optional (we will skip it).

4.4 Eulerian fluid equations in comoving coordinates

We develop here a quasi-Newtonian derivation of perturbation theory. The ultimate justification of the resulting equations, though, requires a full treatment of linear perturbation theory within general relativity.

4.4.1 *Coordinate transformation*

In order to describe the cosmic evolution of non-relativistic matter, but must first consider a general background expansion. In analogy with the case of spherical collapse (though, again, the full justification comes from GR), this requires adding to $d\vec{q}/dt$ the term

$$-\frac{4}{3}m\pi G(\rho + 3p)_{\text{rest}}\,\vec{r}\ , \tag{4.42}$$

where from now we use $\vec{r}$ for the (physical) position. We emphasize that the distribution function and quantities derived from it refer only to matter, and do not include other components (which are assumed to be spatially uniform); as a reminder of this, we will write ρ_{m} instead ρ. The next step is to transform the fluid equations (continuity, Euler, plus Poisson), which we have derived in a Newtonian (fixed) coordinate frame, to comoving coordinates:

$$\vec{x} = \vec{r}/a\ , \qquad d\tau = dt/a\ , \tag{4.43}$$

with a corresponding velocity in these coordinates:

$$\vec{v} \equiv \frac{d\vec{x}}{d\tau} = a\frac{d\vec{x}}{dt} = \frac{d\vec{r}}{dt} - H\vec{r} = \vec{u} - H\vec{r}\ , \tag{4.44}$$

which shows that $\vec{v}$ is precisely the peculiar velocity.

The transformation to comoving coordinates also involves gravity. Instead of the density, we focus on the density pertubation relative to the cosmic mean density, $\Delta\rho_{\text{m}} = \rho_{\text{m}} - \bar{\rho}_{\text{m}}$, or its dimensionless form δ. Similarly, we define a peculiar gravitational potential ϕ_{pec} so that $\phi_{\text{pec}} = 0$ for a homogeneous universe. Indeed, in a uniform universe there is spherical symmetry, so assuming $\phi = \phi(r)$, the Poisson equation (with respect to the proper coordinate $\vec{r}$) is, in spherical coordinates,

$$\nabla_r^2\phi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi G\bar{\rho}_{\text{m}}\ ,$$

where $\bar{\rho}_{\text{m}}$ may depend on time. The solution (assuming $\phi = 0$ at the origin $r = 0$) is $\phi = 2\pi G\bar{\rho}_{\text{m}}r^2/3$. Thus, in general ϕ_{pec} is defined relative to the mean Universe by

$$\phi_{\text{pec}} = \phi - \frac{2\pi}{3}G\bar{\rho}_{\text{m}}r^2\ . \tag{4.45}$$

We note that *mathematically* speaking, the force due to the other components in Eq. (4.42) is equivalent to adding to ϕ a term

$$\frac{2\pi}{3}G(\rho + 3p)_{\text{rest}}\,r^2\ ,$$

since $\vec{\nabla} r^2 = 2\vec{r}$. Thus, the term of Eq. (4.42) does not change any moment equation of the CBE that does not involve ϕ, and it changes the Euler equation by adding, on the right-hand side, the term

$$-\frac{4}{3}\pi G(\rho + 3p)_{\rm rest}\, \vec{r}\ , \tag{4.46}$$

When we transform from coordinates $(\vec{r}, t)$ to $(\vec{x}, \tau)$, we need to also transform the partial derivatives in the fluid equations. Fixed t is the same as fixed τ, so for any function s,

$$\left.\frac{\partial s}{\partial \vec{x}}\right|_{\tau} = a \left.\frac{\partial s}{\partial \vec{r}}\right|_{t} \ . \tag{4.47}$$

A derivative at fixed $\vec{x}$ is more subtle, however. In general,

$$ds = \left.\frac{\partial s}{\partial \vec{r}}\right|_{t} \cdot d\vec{r} + \left.\frac{\partial s}{\partial t}\right|_{\vec{r}} dt\ .$$

Since $\vec{r} = a\vec{x}$, at fixed $\vec{x}$ we have $d\vec{r} = \vec{x}da$. Also, in general, $da/d\tau = a da/dt = a^2 H$. Thus diving df by $d\tau$ at fixed $\vec{x}$ yields

$$\left.\frac{\partial s}{\partial \tau}\right|_{\vec{x}} = aH\vec{r}\cdot \left.\frac{\partial s}{\partial \vec{r}}\right|_{t} + a \left.\frac{\partial s}{\partial t}\right|_{\vec{r}} \ . \tag{4.48}$$

We can also use this in reverse form, as

$$\left.\frac{\partial s}{\partial t}\right|_{\vec{r}} = \frac{1}{a} \left.\frac{\partial s}{\partial \tau}\right|_{\vec{x}} - H\vec{x}\cdot \left.\frac{\partial s}{\partial \vec{x}}\right|_{\tau} \ . \tag{4.49}$$

In what follows, we use short-hand notation

$$\vec{\nabla}_r \equiv \left.\frac{\partial}{\partial \vec{r}}\right|_{t} \ , \qquad \vec{\nabla}_x \equiv \left.\frac{\partial}{\partial \vec{x}}\right|_{\tau} \ .$$

Also, by $\partial/\partial t$ we mean at constant $\vec{r}$, while $\partial/\partial \tau$ implies at constant $\vec{x}$.

4.4.2 *Poisson equation*

Since the Poisson equation is linear in ϕ,

$$\nabla_x^2\,\phi_{\rm pec} = a^2\nabla_r^2\,\phi_{\rm pec} = a^2\nabla_r^2 \left(\phi - \frac{2\pi}{3}G\bar{\rho}_{\rm m} r^2\right) = a^2\left(4\pi G\rho_{\rm m} - 4\pi G\bar{\rho}_{\rm m}\right)\ .$$

Thus, the final result is

$$\nabla_x^2\,\phi = 4\pi G a^2 \bar{\rho}_{\rm m}\delta\ , \tag{4.50}$$

where here and for the rest of this section, we drop the subscript, and use ϕ to denote the peculiar Newtonian potential $\phi_{\rm pec}$.

4.4.3 *Continuity equation*

The continuity equation in the notation of this section is

$$\left.\frac{\partial \rho_{\rm m}}{\partial t}\right|_{\vec{r}} + \vec{\nabla}_r \cdot (\rho_{\rm m}\vec{u}) = 0 \ .$$

The first term is

$$\left.\frac{\partial \rho_{\rm m}}{\partial t}\right|_{\vec{r}} = \left.\frac{\partial}{\partial t}\right|_{\vec{r}} [\bar{\rho}_{\rm m}(1+\delta)] = \frac{d\bar{\rho}_{\rm m}}{dt}(1+\delta) + \bar{\rho}_{\rm m}\left[\frac{1}{a}\left.\frac{\partial \delta}{\partial \tau}\right|_{\vec{x}} - H\vec{x}\cdot\left.\frac{\partial \delta}{\partial \vec{x}}\right|_{\tau}\right] \ ,$$

where we used Eq. (4.49). The second term in the continuity equation is

$$\frac{1}{a}\vec{\nabla}_x \cdot [\bar{\rho}_{\rm m}(1+\delta)(\vec{v}+aH\vec{x})] = \frac{1}{a}\bar{\rho}_{\rm m}\vec{\nabla}_x \cdot [(1+\delta)\vec{v}] + \bar{\rho}_{\rm m}H\left[\vec{x}\cdot\vec{\nabla}_x\,\delta + (1+\delta)\vec{\nabla}_x\cdot\vec{x}\right] \ .$$

We now note that $d\bar{\rho}_{\rm m}/dt = -3\bar{\rho}_{\rm m}H$ (since $\bar{\rho}_{\rm m} \propto a^{-3}$), and $\vec{\nabla}_{\vec{x}}\cdot\vec{x} = 3$. Combining all terms, and dividing by a factor of $\bar{\rho}_{\rm m}/a$, yields:

$$\frac{\partial \delta}{\partial \tau} + \vec{\nabla}_x \cdot [(1+\delta)\vec{v}] = 0 \ . \qquad (4.51)$$

4.4.4 *Euler equation*

The Euler equation can be transformed similarly to the continuity equation. Our starting point is the desired left-hand side, which is the comoving version of the total derivative following a trajectory:

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + \vec{v}\cdot\vec{\nabla}_x \ ,$$

applied to the comoving velocity $\vec{v}$. We apply to it Eq. (4.44), Eq. (4.47), and Eq. (4.48) to obtain

$$\frac{\partial \vec{v}}{\partial \tau} + \left(\vec{v}\cdot\vec{\nabla}_x\right)\vec{v} = aH\vec{r}\cdot\vec{\nabla}_r\,(\vec{u}-H\vec{r}) + a\frac{\partial}{\partial t}\,(\vec{u}-H\vec{r}) + a\left[(\vec{u}-H\vec{r})\cdot\vec{\nabla}_r\right](\vec{u}-H\vec{r}) \ .$$

On the right-hand side, the first term cancels with the second part of the third term, leaving

$$a\frac{\partial u}{\partial t} - a\vec{r}\,\frac{dH}{dt} + a\left(\vec{u}\cdot\vec{\nabla}_r\right)(\vec{u}-H\vec{r}) \ .$$

The last term we split, and then note that $\left(\vec{u}\cdot\vec{\nabla}_r\right)\vec{r} = \vec{u}$. We now note that two of the terms we have here equal a times the left-hand side of the Euler equation in proper coordinates, so we use that equation, with the addition of the term of Eq. (4.46). For simplicity, we assume no anisotropic stress ($\overleftrightarrow{\pi} = 0$). Using also Eq. (4.45), we obtain

$$\left[-\frac{a}{\rho_{\rm m}}\vec{\nabla}_r\,p - a\vec{\nabla}_r\left(\phi + \frac{2\pi}{3}G\bar{\rho}_{\rm m}r^2\right) - \frac{4}{3}\pi Ga(\rho+3p)_{\rm rest}\,\vec{r}\right] - a\vec{r}\,\frac{dH}{dt} - aH\vec{u} \ ,$$

where again we denote $\phi_{\rm pec}$ simply by ϕ. Now we re-write the $\vec{r}$ gradients as $\vec{x}$ gradients using Eq. (4.47), and the remaining $\vec{u}$ back in terms of $\vec{v}$ using Eq. (4.44). We obtain

$$-a\vec{r}\left\{\frac{dH}{dt} + H^2 + \frac{4\pi}{3}G\left[\bar{\rho}_{\rm m} + (\rho+3p)_{\rm rest}\right]\right\} - aH\vec{v} - \frac{1}{\rho_{\rm m}}\vec{\nabla}_x\,p - \vec{\nabla}_x\,\phi \ .$$

Now we use the cosmic acceleration equation for the mean universe to see that

$$\frac{dH}{dt} + H^2 = \frac{d}{dt}\left(\frac{da/dt}{a}\right) + \frac{(da/dt)^2}{a^2} = \frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left[\bar{\rho}_{\rm m} + (\rho + 3p)_{\rm rest}\right] \ .$$

Finally, noting that $H = (da/d\tau)/a^2$, the Euler equation in comoving coordinates is

$$\frac{\partial \vec{v}}{\partial \tau} + (\vec{v} \cdot \vec{\nabla}_x)\vec{v} = -\frac{1}{a}\frac{da}{d\tau}\,\vec{v} - \frac{1}{\rho_{\rm m}}\vec{\nabla}_x\, p - \vec{\nabla}_x\, \phi \ . \tag{4.52}$$

Again, the left-hand side is simply the total derivative following a trajectory, while on the right-hand side, the first term accounts for the redshifting of peculiar velocity ($v \propto 1/a$), the second for the pressure gradient force (when gas pressure is present), and the third for gravity.