Cosmology 2 (Prof. Rennan Barkana): Solution to Homework 2 (Jan. 2023)

a. The circular velocity is given by

$$v_c^2 = \frac{G}{R} \int 2\pi R\,\Sigma(R) dR = 2\pi G \Sigma_0 R_0 \ .$$

Thus,

$$\Omega(R) = \frac{v_c}{R} = \frac{\sqrt{2\pi G \Sigma_0 R_0}}{R} \ ,$$

and $\kappa(R) = \sqrt{2}\,\Omega(R)$.

Corotation resonance:

$$\frac{\omega}{m} = \Omega \ ,$$

which yields:

$$R = \frac{v_c}{\omega} m \ .$$

Lindblad resonances:

$$\frac{\omega}{m} = \Omega \mp \frac{\sqrt{2}\Omega}{m} \ ,$$

which yields:

$$R_\mp = \frac{v_c}{\omega}(m \mp \sqrt{2}) \ .$$

b. The dispersion relation is:

$$(m\Omega - \omega)^2 = \kappa^2 - 2\pi G \Sigma |k| + k^2 c_s^2 \ .$$

Here this gives:

$$\left(m\frac{v_c}{R} - \omega \right)^2 = \kappa^2 - \frac{v_c^2}{R} |k| + k^2 c_s^2 \ .$$

This is a quadratic equation, so there are two solutions for k. We take the + solution, since it has a larger $|k|$, thus also a larger kR at each R, which is what it means to satisfy the tight-winding approximation more accurately. Also, k for this solution is positive so there is no need for the absolute value. Thus,

$$k(R) = \frac{v_c^2}{2c_s^2 R} \left\{ 1 + \sqrt{1 + 4\frac{c_s^2}{v_c^2} \left[\left(m - \frac{\omega R}{v_c} \right)^2 - 2 \right]} \right\} \ .$$

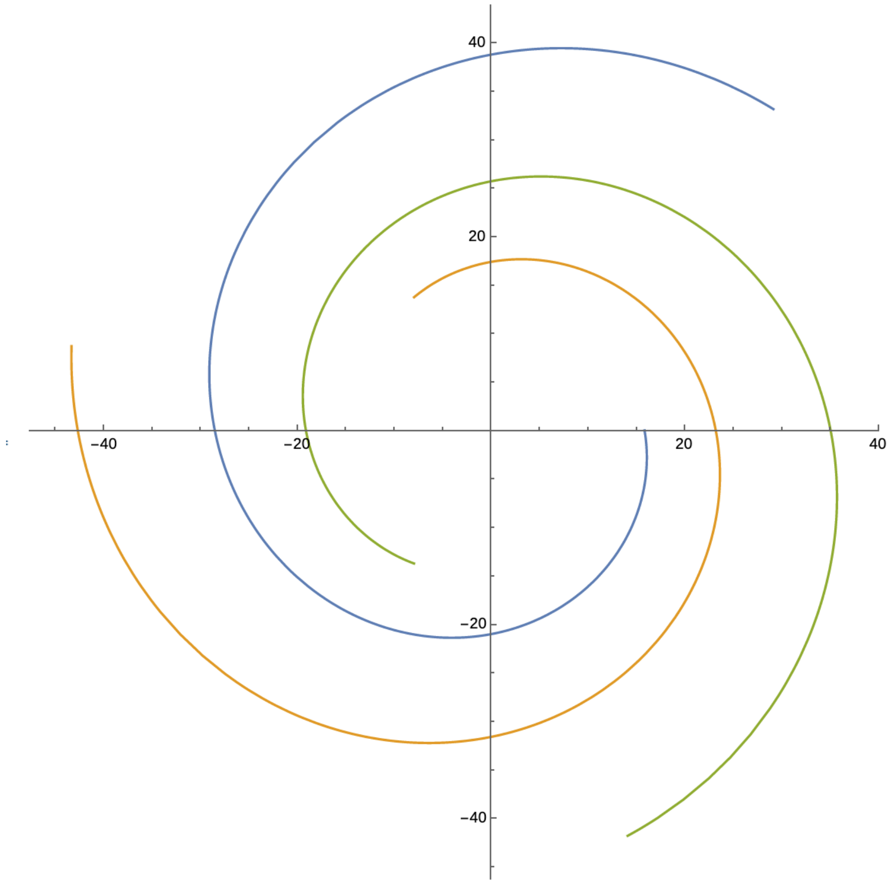
We use this solution between the Lindblad resonances, R_- and R_+ from part **a** above. Thus, the shape function is:

$$f(R) = \int_{R_-}^{R} k(R')dR' \ .$$

The location of a spiral arm in polar coordinates is:

$$\phi = \frac{2\pi}{m} l - \frac{f(R)}{m} \ ,$$

where l goes from 0 to $m-1$. We make a parametric plot of $(x, y) = (R\cos\phi, R\sin\phi)$. Each spiral arm is plotted separately, and then the plots are combined. The first plot below is with the given parameters: $m = 3$, $v_c = 50$ km/s, $c_s/v_c = 0.24$, $v_c/\omega = 10$ kpc. For the second plot we choose $m = 7$, $v_c = 100$ km/s, $c_s/v_c = 0.2$, and $v_c/\omega = 20$ kpc. In these plots, the axes are in kpc.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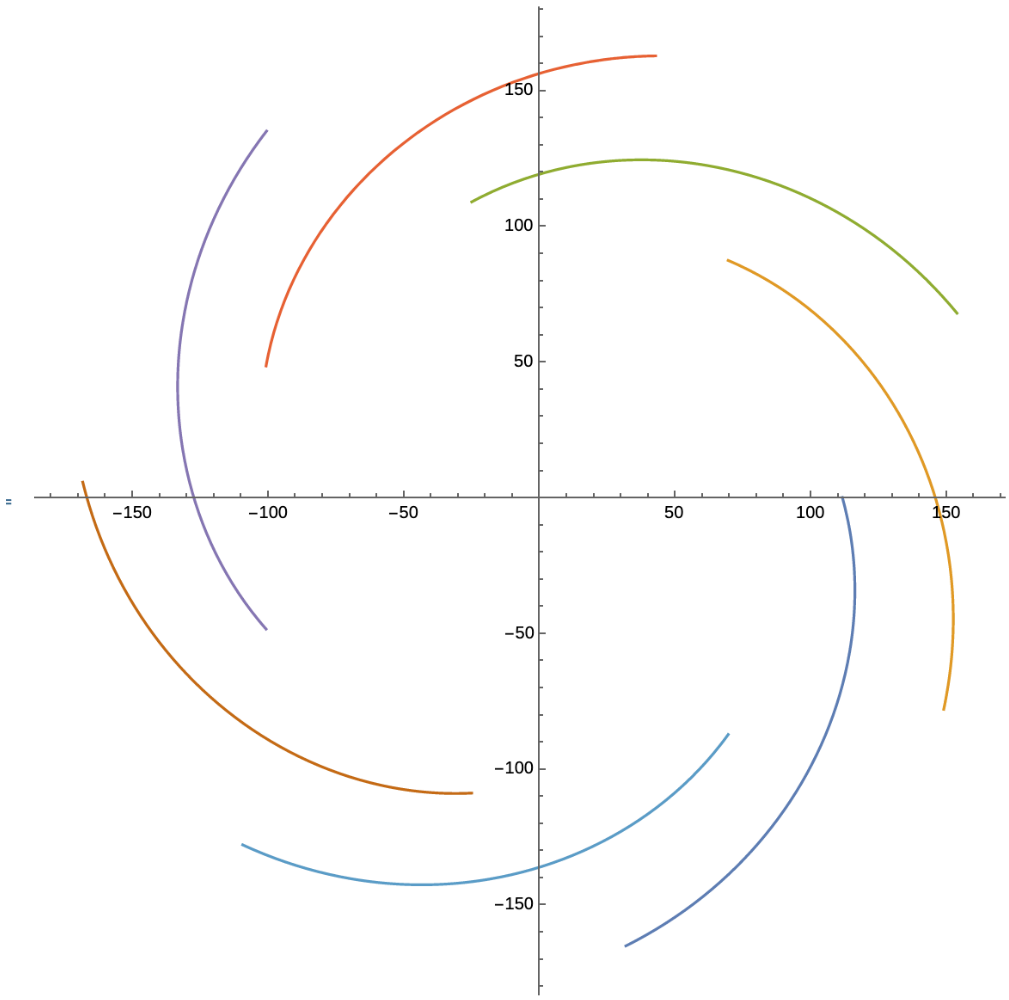
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